THURSDAY, SEPTEMBER 1, 1887.

HIGHER ALGEBRA.

Higher Algebra: a Sequel to Elementary Algebra for Schools. By H. S. Hall, M.A., and S. R. Knight, B.A. (London: Macmillan, 1887.)

ANYONE who imagined from the shortened title of "Higher Algebra," which appears on the back of this volume, that the work extended to that higher region of algebra to which Salmon's well-known "Lessons" are "introductory," would be surprised to find that it contains little beyond what may fairly be regarded as "elementary algebra." Indeed it appears to us that much that is contained in the earlier chapters would have found its place more appropriately in the same authors' "Elementary Algebra for Schools," using their own device of an asterisk to indicate those articles which might, in the case of the ordinary school-boy, be omitted or reserved for a second reading; and thus the awkwardness of breaking up such subjects as ratio, proportion, and progressions into separate parts, by an arbitrary division into lower and higher, would have been avoided.

Apart from this defect of plan, the work before us has great merits as a text-book adapted to the wants of the ordinary student of algebra and to the exigencies of examinations. It is a development and improvement upon "Todhunter," as "Todhunter" was a development and improvement upon "Wood." The main framework is the same: many of the proofs of algebraical theorems have been replaced by better proofs, and new matter has been introduced. Still it remains essentially an artificial framework and has no claim to be regarded as an organic growth from a few central principles, with a corresponding natural relation and affiliation of parts. Thus we find the fundamental laws of algebra for the first time gathered together and discussed in the thirty-fourth chapter (p. 429) of this volume, a chapter of "Miscellaneous Theorems and Examples" for which apparently no fitting place could be found in the framework. It also includes such a fundamental theorem as that known as the "remainder theorem"—that $f(a)$ is the remainder when the rational integral function $f(x)$ is divided by $x - a$—some of its applications, as well as some discussion of symmetrical expressions and identities.

An elementary algebra, written by a master of modern algebraical science in the light of the higher views of the essential nature of algebra which modern investigations have established, and yet with such simplicity that it may be put into the hands of the school-boy, is a desideratum the advent of which is perhaps foreshadowed, though not fully realized, in respect of simplicity at any rate, in Prof. Chrystal's recent work. It would be obviously unfair to criticize the present work from this point of view: our remaining remarks on it, therefore, will be confined to some matters of detail in the order of the chapters of the book itself.

Perhaps the strongest part of the book is the examples, both those which are worked out, and those which are added to each chapter as exercises for the pupil. As far as we have been able to examine them, they are sufficiently numerous, well chosen, and instructive, and also well arranged in each exercise in the order of their difficulty. We are surprised to find in the chapter on "Miscellaneous Equations" that there is no hint or caution given that the root obtained may not satisfy the original equation unless the sign of one or more of the radicals involved in it is changed. In fact, in the example worked out on p. 99, the root $x = 6a$ gives by substitution in the equation $2a - 6a = 4a$! We hold that in all such cases the student should be required to show with what signs of the radicals in the equation each solution is consistent, and what combinations of their signs are impossible; otherwise more than half the instructiveness of the example is lost.

The chapters on ratio and proportion need no remark; but that on variation, as in most books of algebra, is in our opinion unsatisfactory, from the fact that the attention of the student is not called to the distinction between a magnitude and its numerical measure. If A stand for the distance and B for the time, when the speed is uniform, "A varies as B" is a statement true of the magnitudes themselves independent of any particular mode of measuring them; but when from this is deduced the equation $A = mB$, either A and B must be regarded as numerical measures of the distance and time with reference to some particular units, in which case m will have a value depending on the units selected; or else m is a multiplier which, besides altering the numerical value of B into that of A, converts a time into a distance, an extension of the notion of multiplication which, if admitted, ought to be very carefully noted and explained.

After chapters on progressions, we come to one on scales of notation, though there is no reason, apart from the traditional place it has occupied in books on algebra, why such simple questions as are discussed in it, which, if arithmetic were rationally taught, would have been treated in connexion with the theory of decimal numeration and notation, should be regarded as forming a chapter of "Higher Algebra." The algebraical formulæ which encumber this chapter should only be introduced as summing up what has been previously proved in particular instances by direct reasoning from first principles, not in order to prove the propositions themselves.

It would have been well if the chapter on the theory of quadratic *equations* had been made one on that of quadratic *expressions*. By not doing this the opportunity is lost of exemplifying the notion of *continuity* in the changes of such expressions with the change of the variable both in magnitude and sign and their maximum and minimum values, as well as the introduction of the graph (as Prof. Chrystal has done), to illustrate these changes.

The authors state in their preface that the part of algebra which is concerned with permutations and combinations "is made far more intelligible to the beginner by a system of common-sense reasoning from first principles than by the proofs usually found in algebraical text-books," a proposition with which we heartily agree, only that we see no reason why it should be confined to this particular part of algebra.

When we turn, however, to the chapter on permutations and combinations, except that there is a greater variety of proofs, we fail to find any further appeal to

© 1887 Nature Publishing Group

"common-sense reasoning from first principles" than in other text-books. In fact, in some of the proofs the crucial point of the proposition, instead of being elaborated, is so condensed as to make it very difficult to understand, though it is certainly put in a form which may be easily carried into an examination to the perplexity of the examiner, who may well be in doubt whether the examinee who reproduces the words really sees the point of the proof. We hold that the true way of appealing to "common sense" is to take particular cases first, and when these are grasped, the general proof becomes easy. Thus, to find the number of permutations of 4 things (a, b, c, d) taken 3 together, it is plain that the arrangement

$$\overbrace{\overbrace{c\quad d}^{b}\quad\overbrace{d\quad b}^{c}\quad\overbrace{b\quad c}^{d}}^{a}$$

repeated for each of the 4 letters in the top line will give all possible permutations, and that the number is therefore 4 × 3 × 2, and further that the principle of arrangement may be extended to any number of things. This is the essence of the proof given on page 116. It may be said that such exemplification is in the province of the teacher rather than in that of the text-book, but we fear there are many teachers who fail to make things clear in this way to their pupils.

The proof, or rather proofs, of the binomial theorem for positive integral indices are distinct improvements on the cumbrous proof given in Todhunter, the theorem being shown, as it should be, to be a direct consequence of the multiplication of n binomial factors. Euler's proof for any index is carefully stated, and its crucial point emphasized by a preliminary discussion.

Following the binomial theorem comes a chapter on logarithms, which in our opinion would have better followed the chapter on indices in the "Elementary Algebra," as that on interest and annuities might have followed those on progressions. The exponential and logarithmic series would then have followed naturally as a development of the binomial theorem.

The authors have given a chapter on the convergency and divergency of series, in which this important subject is treated with unusual care. We may perhaps demur to the sweeping character of the statement (p. 249) that "the use of divergent series in *mathematical reasoning* leads to erroneous results," but the student cannot be too early or too emphatically warned that a result obtained by the use of divergent series should be verified by other means.

The chapters which follow treat of intermediate coefficients, partial fractions, recurring series, continued fractions, indeterminate equations of the first degree, recurring continued fractions, and indeterminate equations of the second degree, summation of series, the usual elementary theorems of the theory of numbers, the general theory of continued fractions, and probability. All these subjects appear to us to be judiciously and adequately treated, though we should have been glad to see a little more of "common-sense reasoning from first principles" in the elementary chapter on continued fractions, by which it might easily and with advantage have been made to take its place among the chapters of the "Elementary Algebra." In the chapter on summation of series, the authors, as they tell us in the preface, have laid much stress on the "method of differences." As they have gone so far, we think it is a pity that they have not introduced the notation and the elementary propositions of the calculus of differences, which seem to us very naturally to fall within the limits of algebra. In any case, in their use of the symbol Σ they should not have deviated from its proper meaning by making Σn, for instance, include the term n instead of denoting by it the series ending with $n - 1$.

Here the ordinary treatises on algebra end. Our authors have, however, very wisely added a chapter on determinants, containing a satisfactory and sufficient discussion of determinants of the second and third orders, with a useful series of examples of their application, and an indication of the general properties of determinants of any order. The study of this chapter will enable the student to read, without difficulty, treatises on analytical geometry, and afford a good introduction to special works on determinants in general.

Following this is the chapter on miscellaneous theorems and examples, of which we have before spoken, containing a short discussion of the fundamental laws of algebra, then the "remainder theorem," and synthetic division, symmetrical and alternating functions, and elimination.

While the end of ordinary algebra and its various direct applications is undoubtedly a suitable place for a rediscussion of its fundamental laws, as preliminary to the interpretations of double algebra and to the various higher algebras with different fundamental laws, it is strange that our authors have not found the desirability, indeed the necessity, of introducing the other subjects of this chapter, with the exception perhaps of elimination, at a much earlier stage, and as part of a regular sequence in the development of algebraic operations.

The book concludes with a chapter containing the elementary parts of the theory of equations—on the whole judiciously selected. We note it, however, as a defect in this, as in all other treatises we have met with, that Horner's process for approximating to the roots of numerical equations is barely mentioned. We hold that the simplicity and generality of this process is such that it ought to be taught, as a rule (without proof), for finding the roots of numbers, in all treatises on arithmetic, to the exclusion of the cumbrous, uninstructive, and utterly useless method of finding cube roots only, which is usually given; while the proof of the process, which may be made quite easy and intelligible, and its general application to numerical equations, ought to occupy a prominent and early position in every treatise on algebra. Everyone who has made himself expert in the use of Horner's method will, we are sure, agree with us that it gives a power in discussing an algebraical expression with numerical coefficients, which can be obtained in no other way.

R. B. H.

OUR BOOK SHELF.

Outlines of Quantitative Analysis. By A. H. Sexton. (Charles Griffin and Co., 1887.)

It is perhaps as great an evil to err on the side of trying too much as to do too little where more might be done. In this book, intended, as the author tells us, to be put

© 1887 Nature Publishing Group